GENERIC REGULARITY: AREA MINIMIZATION

w/ O. Chodosh, F. Schulze, Z. Wang

- **Question:**

  Given $\Gamma^{n-2} \subseteq \mathbb{R}^n$ closed embedded orientable

  Is there $\Sigma^{n-1} \subseteq \mathbb{R}^n$ compact embedded orientable w/ $\partial\Sigma = \Gamma$

  of least area ("minimizing")?

- **To minimize:** Hypersurfaces ⇝ Federer–Fleming integral currents

- Fleming, Almgren, De Giorgi, Simons, Hardt – Simon

  If $n \leq 7$, yes

  All codimension-1 minimizing integral currents are supported on smooth hypersurfaces

  Full Regularity

Bombieri - De Giorgi - Giusti, Lawson, Simoes

If $n \geq 8$, not necessarily

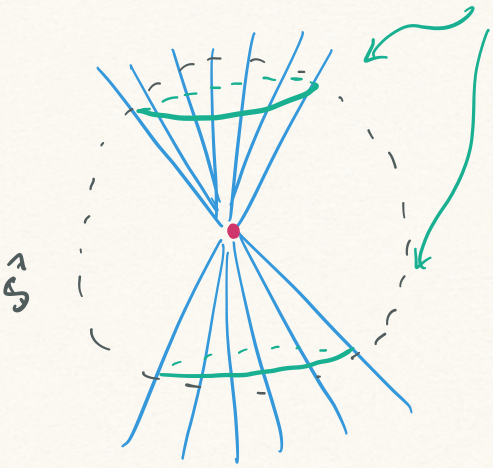
Let $p, q \geq 1$ be s.t. $p+q = n-2$. The "quadratic cone":

$$C_{p,q} = \{ (x,y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} : q|x|^2 = p|y|^2 \}$$

is minimizing (on cpct sets) if:

- $n \geq 9$, or
- $n = 8$ and $(p,q) = (2,4), (3,3), (4,2)$

$\Gamma = S^3(\frac{1}{\sqrt{2}}) \times S^3(\frac{1}{\sqrt{2}})$



$C_{3,3}$ = Simons cone

Federer, Naber - Valtorta, Hardt - Simon

Partial Regularity

Minimizing Σ w/ $\partial\Sigma = \Gamma$ (smooth) satisfy:

- $\dim \operatorname{sing} \Sigma \leq n-8$ (discrete if $n=8$)
- $\operatorname{sing} \Sigma \cap \Gamma = \emptyset$

Hardt-Simon ($\mathbb{R}^8$), C-M-S ($\mathbb{R}^9, \mathbb{R}^{10}$), C-M-S-W ($\mathbb{R}^{11}$)

Generic Regularity

Assume $8 \leq n \leq 11$.

For a Baire-generic set of normal perturbations of $\Gamma^{n-2} \subseteq \mathbb{R}^n$, minimizing $\Sigma^{n-1} \subseteq \mathbb{R}^n$ w/ $\partial\Sigma = \Gamma$ are supported on smooth hypersurface

Remark 1:

In fact our Baire generic set of good Γ is:

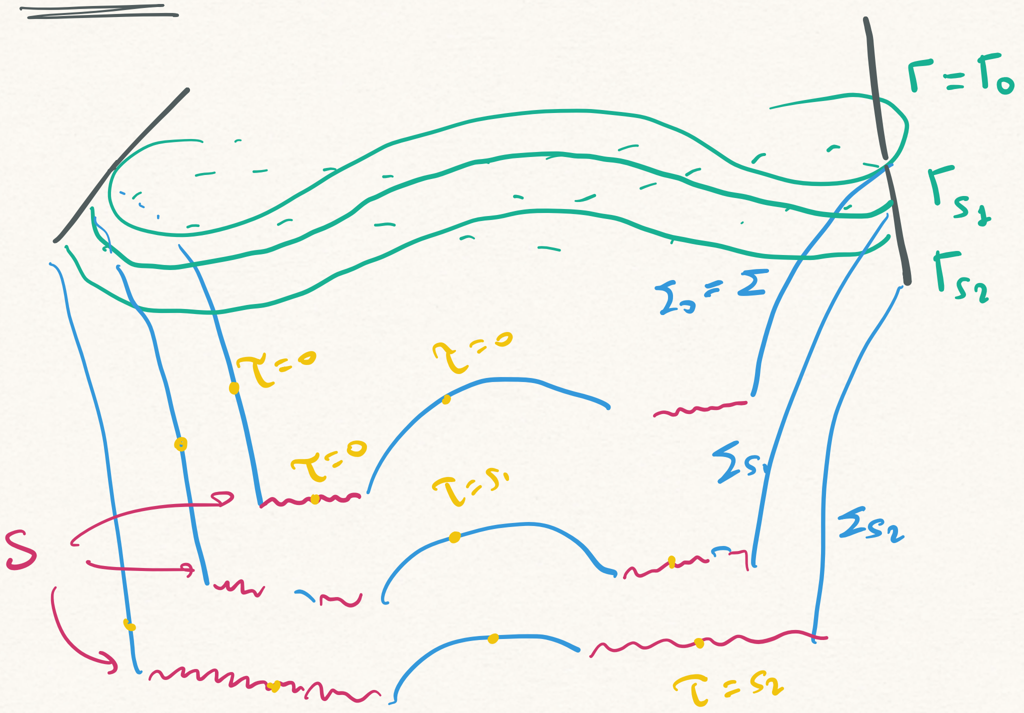
- OPEN: Allard, Hardt-Simon
- DENSE: real ingredient.

Remark 2:

Also holds for area-minimization in $[\alpha] \in H_{n-1}(M, \mathbb{Z})$ for closed oriented M^n with Baire-generic metric g.

..... Smale (M^8), C-M-S (M^9, M^{10}), C-M-S-W (M^{11})

(III) Strategy (for denseness): One-sided perturbations



$$\tau(x) := s \quad \text{for all } x \in \operatorname{supp} \Sigma_s$$

- Construct smooth family $(\Gamma_s)_{0 \le s \le \delta}$, $\Gamma_0 = \Gamma$, moving monotonically along $N\Sigma|_\Gamma$, with unit speed.
- let Σ_s be minimizing for Γ_s.
- Maximum Principle + Monotone $\Rightarrow$ Σ_s pairwise disjoint
- let $S = \bigcup_s \operatorname{sing} \Sigma_s$
- let $\tau : \bigcup_s \Sigma_s \to [0,\delta]$ as above.

~ THEME 1 : Classification of minimizers to one side ~

(Thm) **Hardt - Simon :**

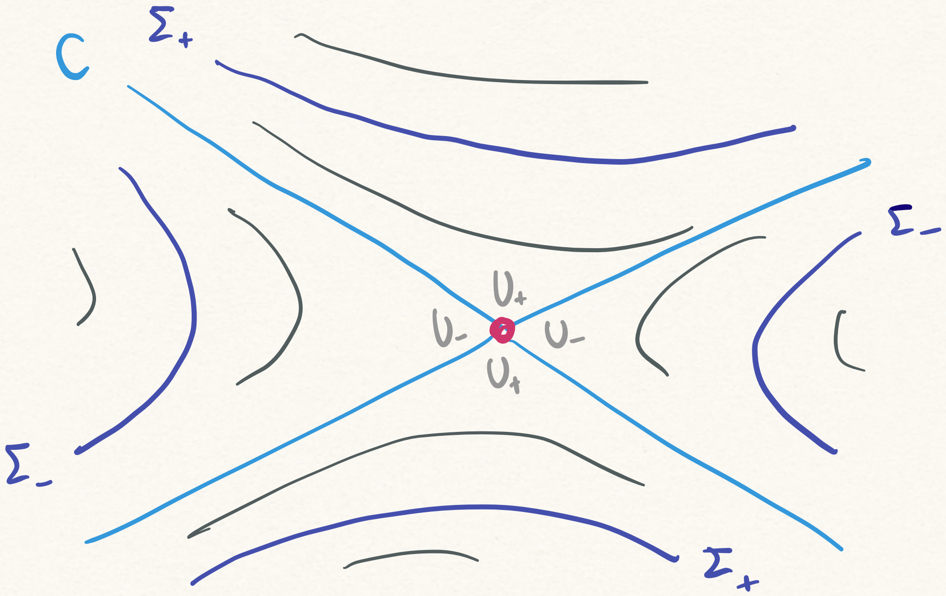
Let $C \subseteq \mathbb{R}^n$ be minimizing cone with $\operatorname{sing} C = \{0\}$

Write $\mathbb{R}^n \setminus C = U_+ \cup U_-$

There exist smooth minimizing $\Sigma_\pm \subseteq U_\pm$ such that

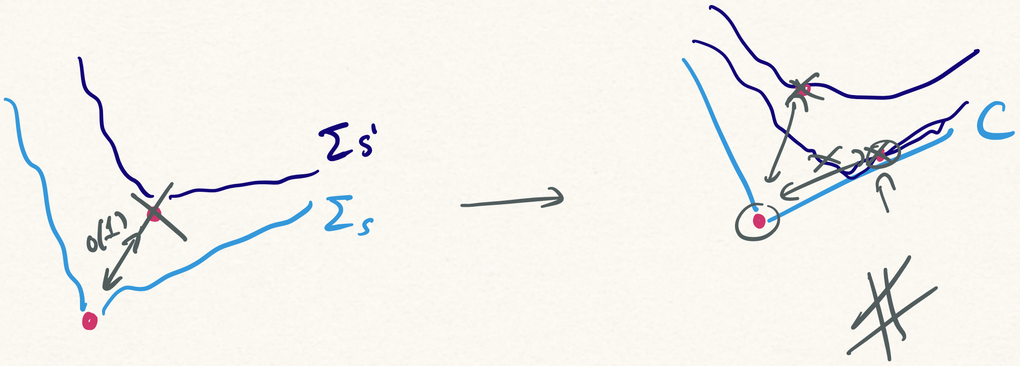
- $\{\lambda \Sigma_\pm\}_{\lambda > 0}$ foliation of $U_\pm$, and moreover
- every minimizing $\Sigma \subseteq \overline{U_\pm}$ must be

$$\Sigma = C \quad \text{or} \quad \lambda \Sigma_\pm \quad \text{for } \lambda > 0.$$



(III) Blow-up argument in $\mathbb{R}^8$

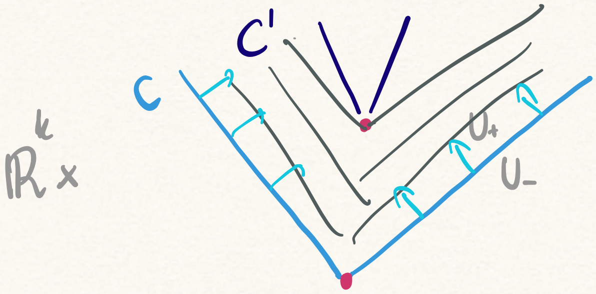
For contradiction assume:



$\Rightarrow$ $\mathcal{S}$ discrete in $\mathbb{R}^8$

- In higher dim we needn't have sing $C = \{0\}$

  but we can always arrange for $\Sigma = $ cone C'



- "Infinitesimal Hardt-Simon"

  Let $C \subseteq \mathbb{R}^n$ be any nonflat minimizing cone

  Write $\mathbb{R}^n \setminus C = U_+ \cup U_-$

  Then minimizing cone $C' \subseteq \overline{U_\pm}$ must be:

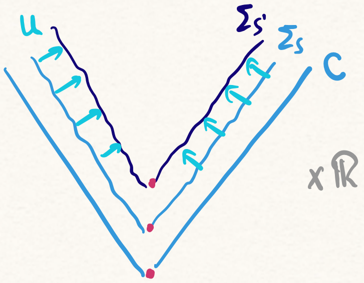
  $$C' = C$$

- Thus:

  Nearby sings of disjoint leaves $\Sigma_s, \Sigma_{s'}$ modeled on $C_s, C_{s'}$

  $\Rightarrow$ $\Sigma_s, \Sigma_{s'}$ modeled on same cone C

  $\Rightarrow$ Normalized height of $\Sigma_{s'}$ over Σ_s produces positive Jacobi field u on underlying C



THEME 2 : Jacobi field analysis

(IIII) Simon, Zhu, Simons

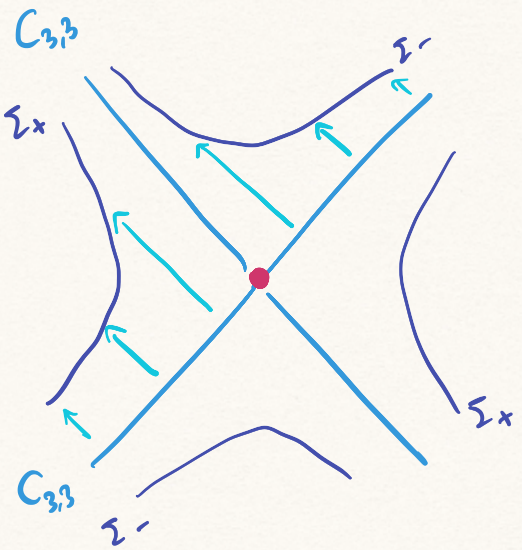
To each minimizing cone $C \subseteq \mathbb{R}^n$ there exists a constant $\alpha(C) \in \mathbb{R}$ such that all positive Jacobi fields on C decay like

$$u \lesssim r^{-\alpha(C)+o(1)}$$

(suitably truncated near sing C). Moreover:

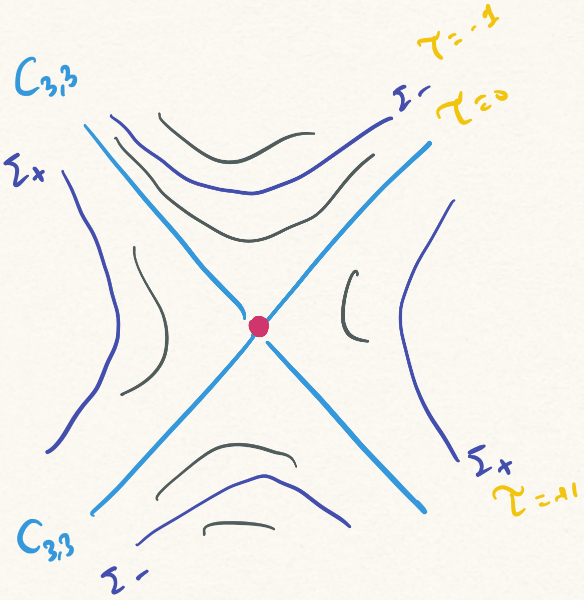
- $\alpha(\text{hyperplane}) = \boxed{0}$
- For all nonflat minimizing cones in $\mathbb{R}^n$
  $$\alpha(C) \geqslant \alpha(\text{quadratic}) = \alpha_n$$
- $\boxed{2} = \alpha_8 > \alpha_9 > \cdots > \boxed{1}$

(III) **Basic Example:** Minimizing leaves on either side of Simons cone $C_{3,3}$



$$\text{Decay} = r^{-\alpha(C_{3,3})} = r^{-2}$$

Reinterpret as Hölder continuity



$$\tau(x) = O(|x|^3) = O(|x|^{2+1})$$

as $x \to 0$

(iii) **Consequence for** $\mathcal{S} = \bigcup_s \operatorname{sing} \Sigma_s$

$$\tau : \bigcup_s \operatorname{supp} \Sigma_s \longrightarrow [0,\delta]$$

At scales where leaves are locally modeled by C

- $\mathcal{S}$ coarsely $\dim(\operatorname{spine} C)$-dimensional
- τ coarsely $(1+\alpha(C))$-Hölder

so, locally:

$$\dim \tau(\mathcal{S}) \leq \boxed{\frac{\dim \mathcal{S}}{1+\alpha(C)} \leq \frac{\dim \operatorname{spine} C}{1+\alpha(C)}}$$

(iiii) In $\mathbb{R}^n$, $n \leq 10$, all nonflat minimizing C have

$$\dim(\operatorname{spine} C) \leq \dim(\operatorname{sing} C) \leq n-8 \leq 2$$

so we can iterate the estimate across scales to get

$$\dim \tau(\mathcal{S}) \leq \boxed{\frac{2}{2+\delta}} < 1$$

i.e. $\boxed{\text{for a.e. } s \in [0,\delta],\ \Sigma_s \text{ is regular}} \Rightarrow \boxed{\text{done}}$

[Partial results : improved Hausdorff bounds on a.e. sing Σ_S
(X. Li) Minkowski —·.——]

- In $\mathbb{R}^{11}$, this only works in scales where $\dim(\operatorname{spine} C) \leq 2$

  but cannot handle $\dim(\operatorname{spine} C) = 3$, i.e.

  $$C = C_0 \times \mathbb{R}^3$$

  for minimizing $C_0 \subseteq \mathbb{R}^{11-3} = \mathbb{R}^8$.

- Can show:

  $$\alpha(C) = \alpha(C_0) = \begin{cases} 2 & C_0 = \text{quadratic} \\ \geq 2+\delta & C_0 \neq \text{quadratic} \end{cases}$$

  So when $C = C_0 \times \mathbb{R}^3$, $C_0 \neq$ quadratic,

  $$\dim \mathcal{T}(S) \leq \frac{3}{1+2+\delta} < 1 \quad \text{as before}$$

- Remaining case: $C = C_0 \times \mathbb{R}^3$, $C_0 =$ quadratic $C_{p,q}$

THEME 3: Refined J.f. analysis

- We have very good understanding of J.f.'s on $C_{p,q}$.

- Two options (after stratification)

(A) Nearby leaves Σ_1, Σ' superlinearly converge to $C_{p,q} \times \mathbb{R}^3$

($\Rightarrow$ strongly unique tangent cone regime)

Then

$$\dim \tau(\mathcal{S}) \le \frac{3}{\boxed{1 + 2 + \delta}} < 1$$

(B) Nearby leaves Σ, Σ' linearly converge to $C_{p,q} \times \mathbb{R}^3$

($\Rightarrow$ possibly nonunique tangent cone)

Then

$$\dim \tau(\mathcal{S}) \le \frac{\boxed{\dim \operatorname{spine}(u)}}{1 + 2} \le \frac{\boxed{2}}{1 + 2} < 1$$

- Themes 2+3 cf. A. Figalli, J. Serra, X. Ros Oton work
obstacle problem generic regularity

Generic regularity: mean curvature flow

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Outline

(1) Background

(2) Generic singularities

(3) Perturbative results via classification of one sided ancient flows

(4) Perturbative results via Jacobi field analysis

Background

Consider $(M^n(t))_{0\leq t<T}$ a smooth mean curvature flow of hypersurfaces in $\mathbb{R}^{n+1}$, i.e.

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = \vec{H} = -H\nu = \Delta_{M(t)}F$$

for a smooth family $F(\cdot, t)$ of parametrisations.

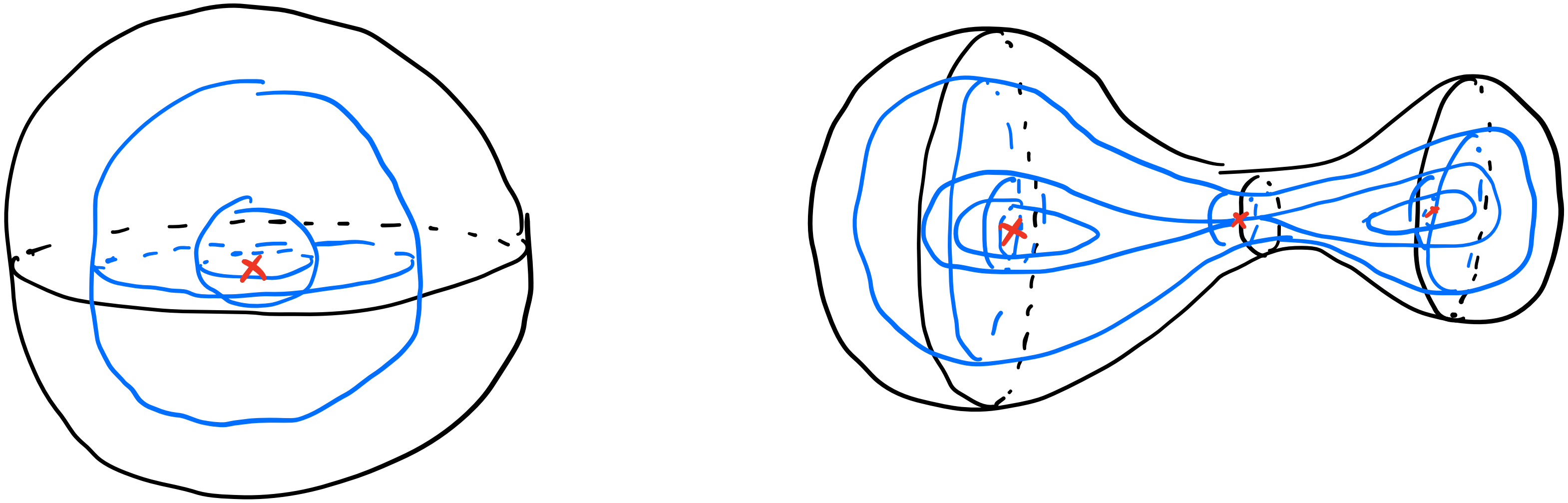
Basic properties:

- Gradient flow of area, geometric heat equation
- Avoidance principle: if $(M_1(t))_{0\leq t<T}$ and $(M_2(t))_{0\leq t<T}$ two solutions of mean curvature flow, then

$$M_1(0) \cap M_2(0) = \emptyset \implies M_1(t) \cap M_2(t) = \emptyset$$

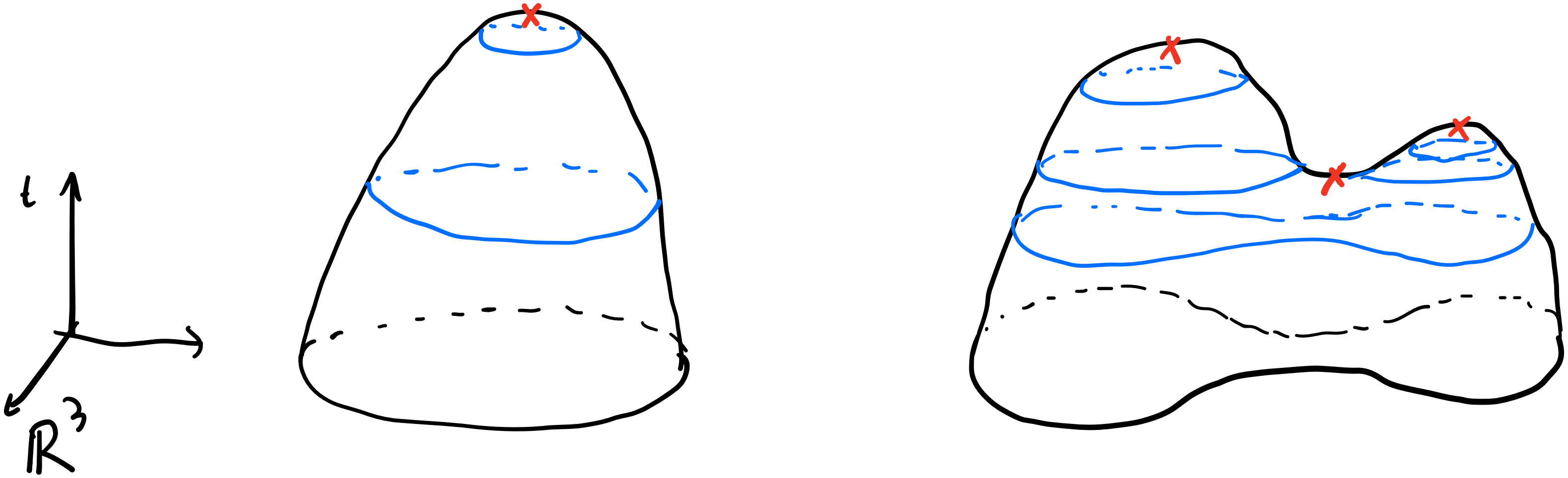
- Finite existence time $\rightsquigarrow$ singularities
- Convexity and mean convexity preserved
- Continuation through singularities as weak mean curvature flow, possibly non-unique

Singularities



Space-time track:

$$\mathcal{M} = \cup_{t\geq 0} M_t \times \{t\} \subset \mathbb{R}^{n+1} \times \mathbb{R}\,.$$



Monotonicity formula: backwards heat kernel based at $X_0 = (x_0, t_0)$:

$$\rho_{X_0}(x,t) = \frac{1}{(2\pi(t_0-t))^{n/2}} e^{-\frac{|x-x_0|^2}{4(t_0-t)}} ,$$

then

$$\frac{d}{dt}\int_{M_t} \rho_{X_0}\, d\mathcal{H}^n \leq -\int_{M_t} \left| \vec{H} + \frac{(x-x_0)^\perp}{2(t_0-t)} \right|^2 \rho_{X_0}\, d\mathcal{H}^n$$

Tangent flows: Consider $\lambda_i \to +\infty$, then subsequentially

$$Pardil_{\lambda_i}(\mathcal{M} - X_0) \rightharpoonup \mathcal{M}' .$$

By monotonicity formula $Pardil_\lambda(\mathcal{M}' \cap \{t<0\}) = \mathcal{M}' \cap \{t<0\}$, i.e.

$$\mathcal{M}'(t) = \sqrt{-t} \cdot \Sigma .$$

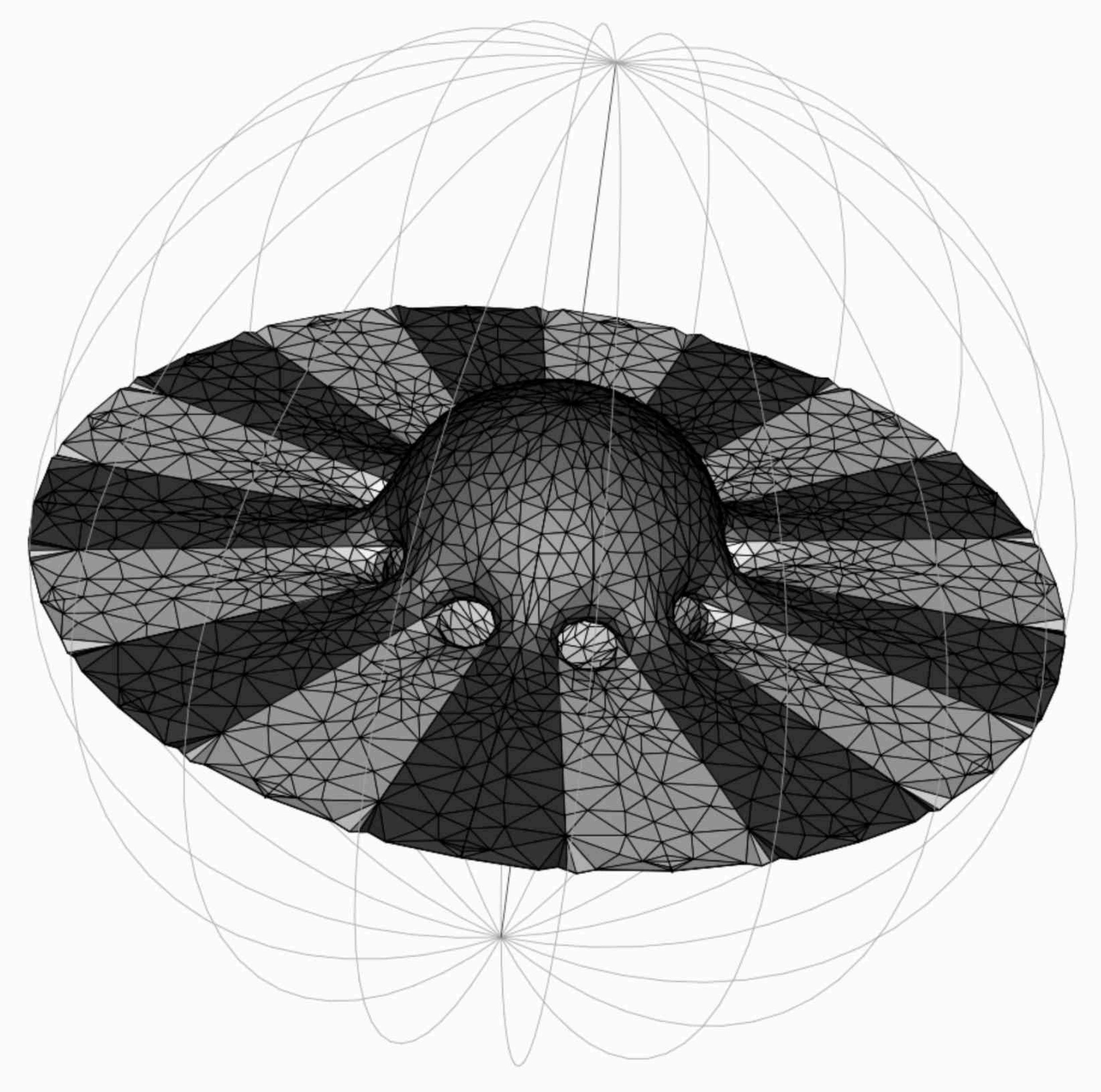
So Σ satisfies

$$\vec{H} = -\frac{x^\perp}{2} .$$

Σ is called a *self-shrinker*.

Examples:

- Plane: $\mathbb{R}^n \subset \mathbb{R}^{n+1}$
- Sphere: $\mathbb{S}^n_{\sqrt{2n}} \subset \mathbb{R}^{n+1}$
- (Generalized) cylinders: $\mathbb{S}^{n-k}_{\sqrt{2(n-k)}} \times \mathbb{R}^k \subset \mathbb{R}^{n+1}$ for $k = 1, \ldots, n-1$
- Huisken ('90): If $H \geq 0$ (preserved under evolution), then these are the only possibilities.
- Angenent ('89): torus of revolution
- Kapouleas-Kleene-Møller ('11), X.H. Nguyen ('11): desingularisation of $\mathbb{R}^2 \cup \mathbb{S}^2_2$



Tom Ilmanen's conjectural shrinker of genus 8 with 9 Scherk handles

(picture used with his permission)

Monotonicity formula and tangent flows

Structure of self-shrinkers:

- $\lim_{\lambda \searrow 0} \lambda \cdot \Sigma = C_\infty$ asymptotic cone (as sets)
- We call Σ *asymptotically conical* if C_∞ and convergence smooth
- L. Wang ('16): $\Sigma^2 \subset \mathbb{R}^3$ embedded with finite genus $\Rightarrow$ Σ^2 has only cylindrical or smoothly conical ends ('16)
- S. Brendle ('16): the only embedded genus zero shrinkers in $\mathbb{R}^3$ are the sphere and the cylinder

Generic singulartities

Fundamental issue:

Zoo of singularities, no hope of classification

Conjecture (Huisken):

A generic mean curvature flow in $\mathbb{R}^3$ has only spherical and cylindrical singularities

Colding-Minicozzi ('12):

- The only linearly stable singularity models are spheres and (generalised) cylinders

Question:

- How to perturb away unstable singularity models?
- Perturb only the initial condition, past singularities?

Perturbative results via classification of one-sided ancient flows

Theorem 1 (CCMS ('20), CCS ('23)): *Let $M^\circ \subset \mathbb{R}^3$ be a closed embedded surface. There exist arbitrary small C^∞ graphs M over M° so that mean curvature flow starting at $M(0) := M$ has only spherical and cylindrical singularities for as long as its singularities have multiplicity one.*

Theorem 2 (CCMS ('20), CCS ('23)): *Let $\Sigma^n \subset \mathbb{R}^{n+1}$, $2 \leq n \leq 6$, be a smooth self-shrinker that is either compact or has only asymptotically conical or cylindrical ends. Up to parabolic dilation around $(0,0) \in \mathbb{R}^{n+1} \times \mathbb{R}$, there exists a unique ancient solution to mean curvature flow $\overline{\mathcal{M}}$ so that $\overline{\mathcal{M}}(t)$ is disjoint from $t \mapsto \sqrt{-t}\,\Sigma$. $\overline{\mathcal{M}}$ is shrinker mean convex: $2H - \langle x, \nu \rangle \geq 0$.*

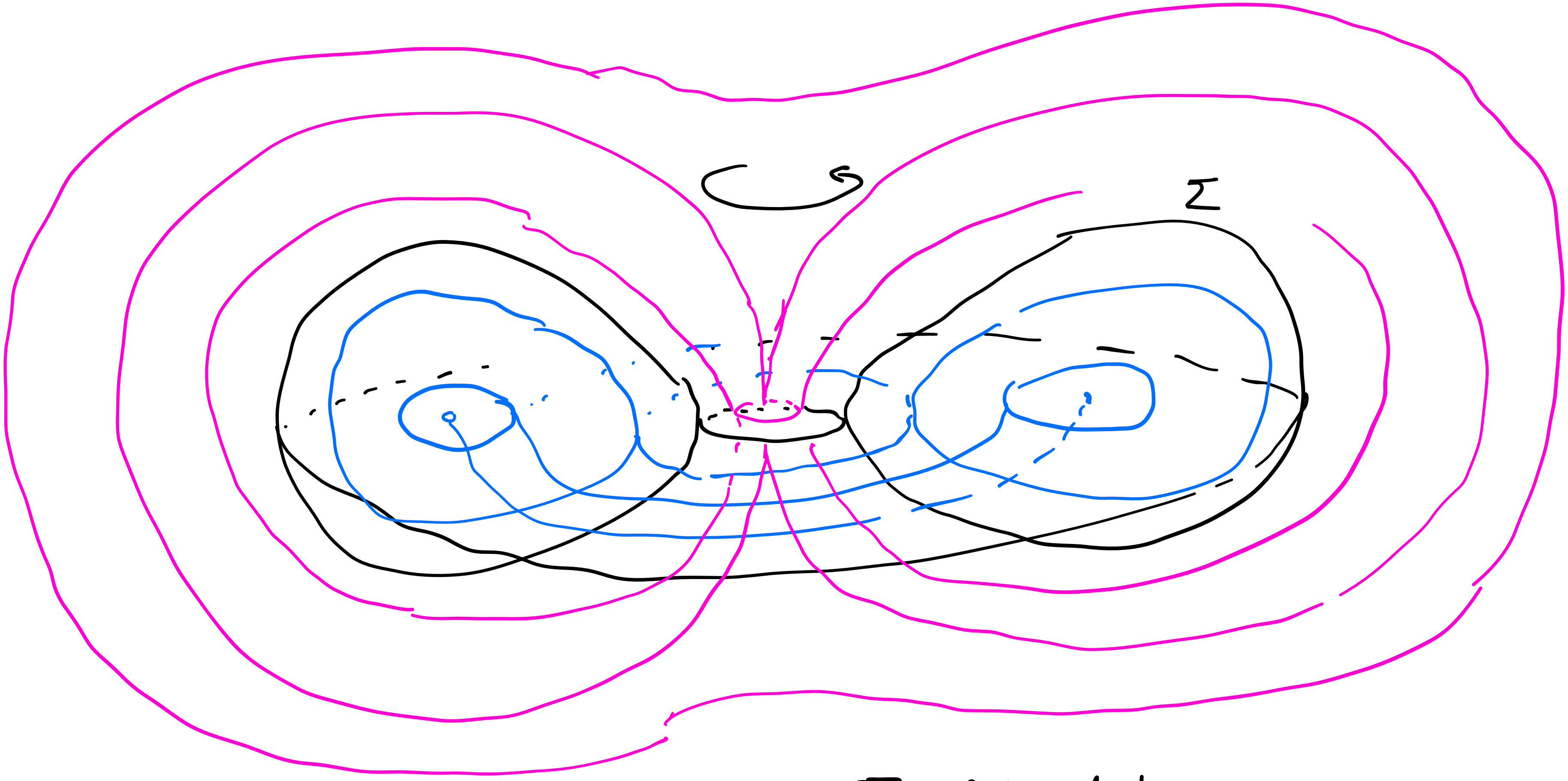
Remark:

- Thm 2 corresponds to Hardt-Simon Liouville theorem.

Strategy of proof of Thm 1: one sided perturbations

- Consider $M_0 \subset \mathbb{R}^{n+1}$ a fixed hypersurface, $\mathcal{M}_0$ a weak mean curvature flow starting at M_0.
- Consider a foliation $\{M_s\}_{s\in(-1,1)}$ around M_0. Embedd the flow $\mathcal{M}_0$ into a family of (weak) flows $\mathcal{M}_s$ starting at M_s.
- Avoidance principle: $\mathcal{M}_s(t) \cap \mathcal{M}_{s'}(t) = \emptyset$ for $s \neq s'$.
- Consider (x_0, t_0) a singular point of $\mathcal{M}_0$ and $\lambda_i \to \infty$ such that $ParDil_{\lambda_i}(\mathcal{M}_0 - (x_0, t_0)) \rightharpoonup \mathcal{M}'$, a tangent flow at X.
- Pass the whole foliation to the limit simultaneously, i.e. consider the flows $ParDil_{\lambda_i}(\mathcal{M}_s - (x_0, t_0))$ as $\lambda_i \to \infty$.
- Choosing $s_i \searrow 0$ carefully as $\lambda_i \to \infty$, up to a subsequence, $ParDil_{\lambda_i}(\mathcal{M}_{s_i} - (x_0, t_0))$ will converge to a non-empty flow $\overline{\mathcal{M}}$ that stays on one side of the original tangent flow $\mathcal{M}'$ and is ancient.
- Thm 2 $\Rightarrow$ $\overline{\mathcal{M}}$ is unique up to parabolic scaling, moves in a rescaled sense in one direction. $\Rightarrow$ thus has only spherical and cylindrical singularities and has genus zero near $(0, 0)$.
- Use this to find a choice of s small so that $\mathcal{M}_s$ has only spherical and cylindrical singularities near (x_0, t_0) and strictly drops genus.
- Iterate.

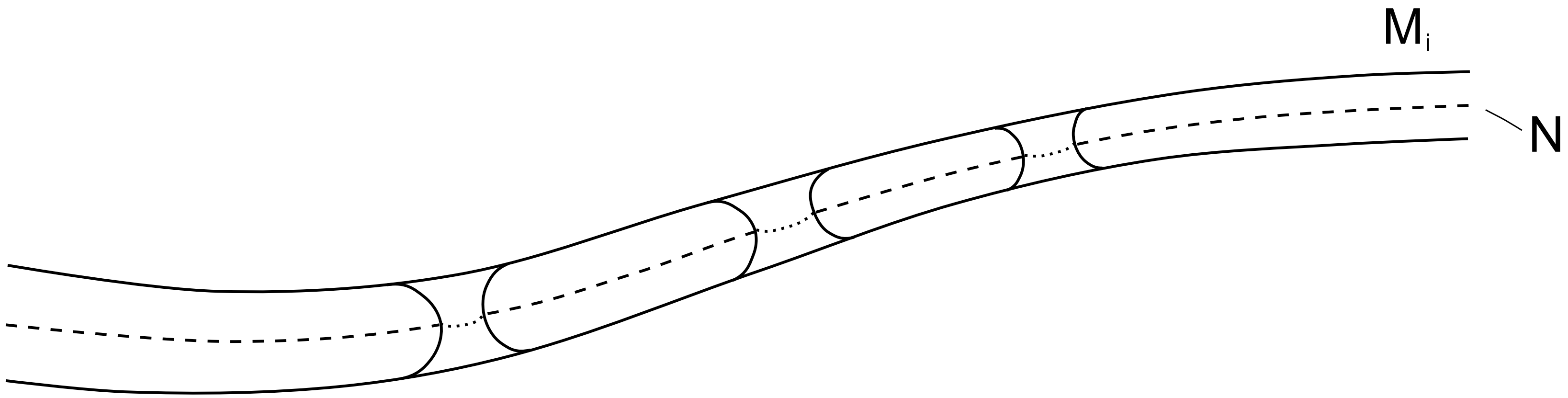
Genus drop: Angenent torus



Σ drilled torus

$\tilde{M}_1(t) = \frac{1}{\sqrt{-t}} \bar{M}_1(t)$ $\hat{M}_2(t) = \frac{1}{\sqrt{-t}} \bar{M}_2(t)$

Multiplicity



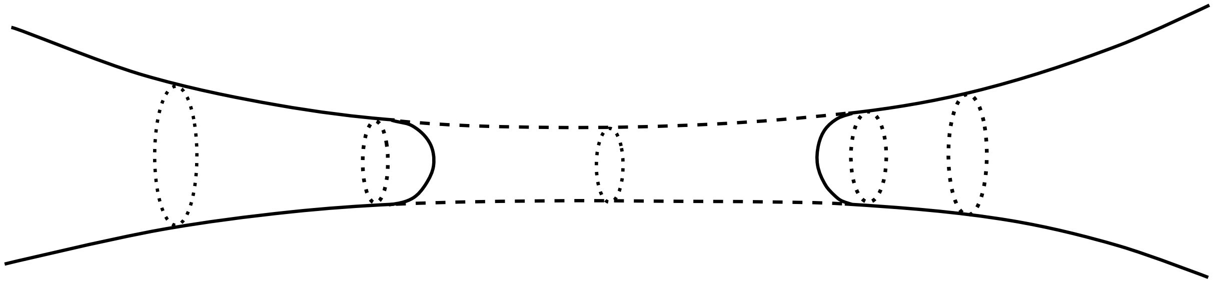
Convergence of the surfaces M_i with multiplicity two to the dotted surface N, while "necks" are pinching off.

Theorem (Bamler – Kleiner ('23)): *For closed embedded surfaces* $M(0) \subset \mathbb{R}^3$*, mean curvature flow has only singularities with multiplicity one at the first non-generic time.*

Corollary: *Let* $M^\circ \subset \mathbb{R}^3$ *be a closed embedded surface. There exist arbitrary small* C^∞ *graphs* M *over* M° *so that mean curvature flow starting at* $M(0) := M$ *has only multiplicity one spherical and cylindrical singularities.*

Flows with surgery

Surgery:



- Close to a neck-pinch singularity, replacing a cylindrical piece by two spherical caps.
- Surgery for mean curvature flow of 2-convex surfaces (Huisken-Sinestrari ('09), Haslhofer-Kleiner ('17), Brendle-Huisken ('16, '17))

Theorem (Daniels-Holgate ('21)): *Any (weak) mean curvature flow with only spherical and neck-pinch singularities starting from a smooth closed embedded hypersurface* $M^n \subset \mathbb{R}^{n+1}$ *can be approximated by smooth flows with surgery.*

Corollary: *Let* $M^\circ \subset \mathbb{R}^3$ *be a closed embedded surface. There exist arbitrary small* C^∞ *graphs* M *over* M° *and a smooth mean curvature flow with surgery starting from* M*.*

Entropy (Colding-Minicozzi)

$$\lambda(M) := \sup_{x_0 \in \mathbb{R}^{n+1}, t_0 > 0} \int_M (4\pi t_0)^{-n/2} e^{-\frac{|x-x_0|^2}{4t_0}} \, d\mu$$

- $t \mapsto \lambda(M_t)$ monotonically decreasing under mean curvature flow

Classification of surfaces of low entropy:

- $\lambda(\mathbb{S}^n) < \lambda(\mathbb{S}^{n-1} \times \mathbb{R}) < \cdots < \lambda(\mathbb{S}^1 \times \mathbb{R}^{n-1})$
- Colding-Ilmanen-Minicozzi-White ('13): the round sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ has lowest entropy among non-planar self-shrinkers.
- Bernstein-Wang ('16) (Zhu ('20)): the round sphere has lowest entropy among all closed hypersurfaces.
- Bernstein-Wang ('18): $\mathbb{S}^1 \times \mathbb{R}$ has second least entropy among non-planar self-shrinkers in $\mathbb{R}^3$.
- Bernstein-Wang ('18): $M^3 \subset \mathbb{R}^4$ closed and $\lambda(M) \leq \lambda(\mathbb{S}^2 \times \mathbb{R}) \Rightarrow M$ diffeomorphic to $\mathbb{S}^3$.

Perturbative results with low entropy: Jacobi field analysis

Theorem 2 (CMS ('23)): *If $M^3 \subset \mathbb{R}^4$ is a closed embedded hypersurface with entropy $\lambda(M) \leq 2$ then there exist arbitrarily small C^∞ graphs M' over M such that the mean curvature flow starting from M' has only multiplicity-one singularities of $\mathbb{S}^3, \mathbb{S}^2 \times \mathbb{R}$, and $\mathbb{S}^1 \times \mathbb{R}^2$-type.*

Remark:

- Uses separation estimates at small scales via parabolic Jacobi field analysis.
- Works also in $\mathbb{R}^5$.
- Entropy assumption is mainly needed to rule out higher multiplicity.

Low entropy Schoenflies theorem

Schoenflies conjecture:

Any smoothly embedded $S^3 \subset \mathbb{R}^4$ bounds a smooth 4-ball.

Corollary (CCMS ('20, '21)): *If* $M^3 \subset \mathbb{R}^4$ *has entropy* $\lambda(M) \leq \lambda(\mathbb{S}^2 \times \mathbb{R})$, *then after a small* C^∞*-perturbation to a nearby hypersurface* M', *the mean curvature flow* $M'(t)$ *is completely smooth until it disappears in a round point.*

This yields an alternate proof of the low entropy Schoenflies theorem of Bernstein-Wang:

Theorem (Bernstein-Wang ('20)): *If* $M^3 \subset \mathbb{R}^4$ *has entropy* $\lambda(M) \leq \lambda(\mathbb{S}^2 \times \mathbb{R})$ *then* M *is smoothly isotopic to the round* $\mathbb{S}^3$.

Combining Theorem 2 with the approximation result by Daniels-Holgate we can improve this further:

Corollary (CCMS, Daniels-Holgate ('21)): *Any smoothly embedded* $M^3 \subset \mathbb{R}^4$ *which is homeomorphic to* $\mathbb{S}^3$ *and has entropy* $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$ *is smoothly isotopic to the round* $\mathbb{S}^3$.

Thank you!